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AN IDEAL "DOUBLE-IMPULSE" MODEL OF A CLOCK WITH AN ANALYTIC STRONGLY ISOCHRONOUS NATURAL-MODE OSCILLATOR†

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A "double-impulse" model of a clock with a single degree of freedom and one counter impulse and one pushing impulse in the period is constructed. A non-linear oscillator with a "natural period", which is analytic with respect to damping and a restoring force, is considered as the oscillatory system of the clock. In the model proposed, the trajectory which "enters" the stationary point after a finite time plays a decisive role in the possibility of realizing the hard mode of self-excited oscillations which is customary in the case of real clocks. © 1998 Elsevier Science Ltd. All rights reserved.

It is well known [1, 2] that any clock mechanism consists of three units: an oscillator in the form of a pendulum or a balance wheel, a winding mechanism, which is the source of energy, and an escapement unit which connects the winding mechanism with the oscillator and transmits the appropriate impulses to the latter. One of the principal requirements imposed on the construction of modern clock mechanisms with an oscillator which has a "natural period" is the requirement of isochronism of the natural modes of the oscillator which enable oscillations of possibly greater amplitude to be maintained in the running of the clock which are less susceptible to the effect of external dynamic actions [2].

Types of non-linear dynamic models of clocks with impulses, which have not been considered previously, when the oscillator is analytic and has a "natural period", that is, oscillations can be executed when the escapement unit is disconnected, are constructed below. Such clock models can be used to develop new designs for mechanism which ensure the accuracy in the running of a clock with a hard mode of self-excited oscillations.

1. A LIÉNARD OSCILLATOR OF DAMPED NATURAL MODES

The equation of motion of an oscillator with a "natural period" and a single degree of freedom, which is used in clock models, is given in general form by the relation

$$\ddot{x}+f(x,\ \dot{x})+g(x)=0,\ \ g(x)=x+g_0(x)$$

We shall concentrate on the treatment of an oscillator called a Liénard oscillator which is described by the equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \tag{1.1}$$

where the non-linear "coefficient" of friction $f(x)$ and the non-linear component of the restoring force $g_0(x)$ are holomorphic in any range of variation of the independent variable x of the function such that $f(0)=0, f(x)>0$ when $x\neq 0$ and $g_0(0)=0$ and $xg(x)>0$ when $x\neq 0$.

With these assumptions, the sole finite stationary point O (0, 0) of the corresponding oscillator (1.1) of the dynamical system

$$\dot{x}=y,\ \dot{y}=-g(x)-f(x)y \tag{1.2}$$

will be the stable focus.

With respect to this focus, we shall assume that the spirals, along which the representative points move in a clockwise direction as t increases, completely fill the whole of the phase plane xOy.

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It has been noted above that one of the fundamental demands made on clock mechanisms involves the requirement that the natural modes of the oscillator should be isochronous [3].

If we introduce the idea of a strongly isochronous oscillator, for which the "half-period" of the vibrations in the half-plane $x \geqslant 0$ of the phase plane is the same as the "half-period" of the oscillations in the half-plane $x \leqslant 0$ of the phase plane, it is then obvious that the accuracy of the running of a clock with such an oscillator can only be increased.

The mathematical aspects of this issue are as follows. If one makes use of the previously proposed mode of reasoning [4] and one of the results [5, Theorem 4], it can then be shown that the following theorem holds.

Theorem 1.1 In order that a high degree of isochronism of the oscillations should occur in the case of the dynamical system

$$\dot{x} = -y - X(x, y), \quad \dot{y} = x + Y(x, y)$$

where the functions X and Y, which are holomorphic in the neighbourhood of the point O (0, 0), do not contain free and linear terms, the existence of a unique transformation

$$u = x, \quad v = y + \sum_{k+l=2}^{\infty} \beta_{kl} x^k y^l \tag{1.3}$$

which converts the dynamical system being considered into a system of the form

$$\dot{u} = -v + u \sum_{s+v=1}^{\infty} \gamma_{sv} \, u^s v^v, \quad \dot{v} = u + v \sum_{s+v=1}^{\infty} \gamma_{sv} \, u^s v^v \tag{1.4}$$

is necessary and sufficient.

It can then be immediately verified that, if

$$x^3 g_0(x) = \left[\int_0^x s f(s) ds \right]^2 \tag{1.5}$$

then a change of the time $t = -t$ and a transformation of the form of (1.3)

$$u = x, \quad v = y + x\sqrt{g_0(x)/x}$$

converts the corresponding oscillator (1.1) and the dynamical system (1.2) into a system of the form of (1.4)

$$\dot{u} = -v + u\sqrt{g_0(u)/u}, \quad \dot{v} = u + v\sqrt{g_0(u)/u}$$

In accordance with Theorem 1.1, the last statement means that the oscillator (1.1) is strongly isochronous, subject to condition (1.5).

We note here that, in accordance with Theorem 1.1, condition (1.5) is not only sufficient but also a necessary condition for the strong isochronism of the oscillations of a Liénard oscillator (1.1).

We will now investigate certain other properties of oscillator (1.1).

We consider an arbitrary spiral S from the family of spiral-trajectories of system (1.2), which leaves from an arbitrary point on the positive semi-axis of the ordinate in the phase plane and completes a single turn around the stationary point. Suppose that the spiral intersects the positive and negative semi-axes of the abscissa in the phase plane at the points ξ and η, respectively and that

$$\begin{aligned} &y_\xi = y(x, \xi), \quad z_\xi = z(x, \xi), \text{ where } 0 \le x \le \xi \\ &z_\eta = z(x, \eta), \quad y_\eta = y(x, \eta), \text{ where } \eta \le x \le 0 \end{aligned} \tag{1.6}$$

are the equations of the arcs of the spiral S located in the first, fourth, third and second quadrants, respectively.

Next, suppose that

$$\Phi(x,\ \xi)=y(x,\ \xi)+z(x,\ \xi),\quad \Phi(x,\ \eta)=y(x,\ \eta)+z(x,\ \eta)$$
$$\Phi(\xi,\ \xi)=\Phi(\eta,\ \eta)=0$$

Lemma 1.1. The inequalities

$$\Phi(x,\ \xi)>0 \text{ when } 0\le x<\xi,\quad \Phi(x,\eta)<0 \text{ when } \eta<x\le 0 \tag{1.7}$$

hold.

Proof. We will prove the validity of the first inequality of (1.7). The feasibility of the second inequality is proved in a similar way. When account is taken of the notation adopted, we have

$$\frac{d\Phi(x,\ \xi)}{dx}=-2f(x)-\frac{g(x)}{y_\xi z_\xi}\Phi(x,\ \xi),\quad 0\le x\le\xi \tag{1.8}$$

According to the assumption that $f(x) > 0$ when $x \neq 0$, it therefore follows from (1.8) that subintervals cannot exist in the interval $(0, \xi]$ in which $\Phi(x, \xi) \equiv 0$. Bearing this in mind, we shall assume that the first of inequalities (1.7) is not satisfied. Then, by virtue of the preceding discussion, this will mean that an interval $[\xi_1, \xi_2] \subset (0, \xi]$ must necessarily exist such that

$$\Phi(x,\ \xi)<0 \text{ when } \xi_1\le x<\xi_2 \text{ and } \Phi(\xi_2,\xi)=0 \tag{1.9}$$

lies in the interval $(0, \xi)$.

On integrating Eq. (1.8) within the limits from ξ_1 to ξ_2 and taking account of (1.9), we arrive at the inequality $\Phi(\xi_2, \xi) - \Phi(\xi_1, \xi) < 0$ from which it follows that $\Phi(\xi_1, \xi) > 0$. But the inequality obtained contradicts relation (1.9). Hence, the function $\Phi(x, \xi)$ is strictly positive when $0 < x < \xi$.

In order to complete the proof it remains to show that $\Phi(0, \xi) > 0$. Actually, if this inequality is not satisfied, it is obvious that $\Phi(0, \xi) = 0$ and, then, by choosing a sufficiently small number $\varepsilon(0 < \varepsilon < \xi)$ and integrating Eq. (1.8) from 0 to ε, we arrive at the equality

$$\Phi(\varepsilon,\ \xi)=\frac{1}{h(\varepsilon,\ \xi)}(\Phi(0,\ \xi)-2\int_0^\varepsilon f(x)h(x,\ \xi)dx),\quad h(x,\ \xi)=\exp\left(\int_0^x\frac{g(s)}{y_\xi z_\xi}ds\right)$$

which, when account is taken of the fact that at $\Phi(0, \xi) = 0$, leads to the impossible inequality $\Phi(0, \xi) \leqslant 0$.

This proves the lemma.

Now suppose that

$$H_1(x,\ \bar\xi,\ \xi)=y(x,\ \xi)-y(x,\ \bar\xi),\quad H_2(x,\ \bar\xi,\ \xi)=z(x,\ \bar\xi)-z(x,\ \xi),\ \text{where } 0\le x\le\bar\xi\le\xi$$
$$H_1(x,\ \eta,\ \bar\eta)=y(x,\ \eta)-y(x,\ \bar\eta),\quad H_2(x,\ \eta,\ \bar\eta)=$$
$$=z(x,\ \bar\eta)-z(x,\ \eta), \text{where } \eta\le\bar\eta\le x\le 0$$

Lemma 1.2. The inequalities

$$\Delta_1(x)\triangleq H_1(x,\ \bar\xi,\ \xi)-H_2(x,\ \bar\xi,\ \xi)>0 \text{ when } 0\le x\le\bar\xi\le\xi$$
$$\tag{1.10}$$
$$\Delta_2(x)\triangleq H_1(x,\ \eta,\ \bar\eta)-H_2(x,\ \eta,\ \bar\eta)<0 \text{ when: } \eta<\bar\eta\le x\le 0$$

hold.

Proof. We will prove that the first of inequalities (1.10) holds. The feasibility of the second inequality is proved in the same way. Thus, at the point $x = \bar\xi$, the left-hand side of the first relation in (1.10), represented in the form $\Delta_1(\bar\xi) = y(\bar\xi, \xi) + z(\bar\xi, \xi)$, is strictly positive, by Lemma 1.1. It follows from the continuity of the functions H_1 and H_2 with respect to the variable x that the left-hand side of the first relation of (1.10) is also strictly positive for $x < \bar\xi$ only if the difference $\bar\xi - x$ is sufficiently small. Then, if the first of inequalities (1.10) does not hold in the interval $0 \leqslant x \leqslant \bar{\xi} < \xi$ a point x^* must exist, lying to the left of $\bar{\xi}$, such that $\Delta_1(x) > 0$ when $x^* < x \leqslant \bar{\xi}$ but $\Delta_1(x^*) = 0$.

We have

$$\frac{d\Delta_1(x)}{dx} = g(x)\left(\frac{H_1}{y_\xi y_{\bar{\xi}}} - \frac{H_2}{z_{\bar{\xi}} z_\xi}\right) < g(x)\frac{\Delta_1(x)}{y_\xi y_{\bar{\xi}}} \tag{1.11}$$

where $0 \leqslant x < \bar{\xi}$ since, by Lemma 1.1, $|y_\xi| > |z_\xi|$ and, consequently, $y_\xi y_{\bar{\xi}} > z_{\bar{\xi}} z_\xi$.

Integrating inequality (1.11) from $x^* + \varepsilon$ to x, where $\varepsilon > 0$ and $x^* < x < \bar{\xi}$, we obtain the inequality

$$\ln\left|\frac{\Delta_1(x)}{\Delta_1(x^* + \varepsilon)}\right| < \int_{x^*+\varepsilon}^{x} \frac{g(s)}{y_\xi y_{\bar{\xi}}} ds$$

which becomes contradictory when $\varepsilon \to 0$ by virtue of the boundedness of the right-hand side. The resulting contradiction proves the lemma.

2. A MODEL OF A CLOCK WITH COUNTER AND PUSHING IMPULSES

Consider a clock mechanism with a strongly isochronous Liénard oscillator (1.1). We will assume that the escapement transmits instantaneous impulses to the oscillator by means of a counter impulse in a direction opposite to the motion of the oscillator, leading in its interpretation in the phase plane to an instantaneous increase in velocity at a certain constant value of $L_1 > 0$ and a pushing impulse in the direction of motion of the oscillator which leads to an instantaneous increase in the velocity at a constant value of $L_2 > L_1$. It is assumed here that the transmission of the impulse (jump) is only performed once in the equilibrium position at a non-zero rate of change in the amplitude, that is, at the instant when $x = 0, y \neq 0$.

In this case the mathematical model can be written in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y + \frac{1}{2}(L_1 + L_2(L_2 - L_1)sign(y))\delta(x) \end{aligned} \tag{2.1}$$

where $\delta(x)$ is the Dirac delta-function.

The subsequent investigations involves a consideration of the possible types of motions in the case of pulsed system (2.1).

3. PERIODIC OSCILLATIONS OF SYSTEM (2.1) WITH A SINGLE JUMP IN THE HALF-PERIOD OF THE OSCILLATION PERIOD OF THE OSCILLATOR

Suppose that

$$H(x, \xi) = y(x, \xi) - z(x, \xi), \text{ where } 0 \leq x \leq \xi$$

Then, by virtue of the property of the continuous dependence of the solutions of the initial equations on the initial data and the fact that the spiral-trajectories of system (1.2) completely fill the phase plane, it can be confirmed that the function $H(0, \xi)$, where $\xi \geqslant 0$, increases strictly monotonically when $\xi \to +\infty$ and that $H(0, \xi) \to +\infty$. Hence, a unique solution ξ_0 of the algebraic equation $L_1 = H(0, \xi)$, defined by the equality $\xi_0 = H^{-1}(L_1)$, exists for any value of $L_1 > 0$. This solution obviously corresponds to periodic oscillations of system (2.1) with a single jump in a half-period of the oscillation period of the oscillator to which, in turn, there are corresponds an orbitally asymptotic stable limiting cycle.

We shall now prove the last point. We put $y_0 = y(0, \xi_0)$, $z_0 = z(0, \xi_0)$, assuming, for brevity, that the symbols y_0, z_0 (as well as symbols similar to them later) denote points on the $x = 0$ axis with values on the ordinates corresponding to the symbols.

Suppose that, using this notation, a contour

$$(y_0, \xi_0, z_0 \xrightarrow{L_1} y_0)$$

corresponds to periodic oscillations in the phase plane, that is, a discontinuous trajectory of system (2.1) which passes through the points y_0, ξ_0, z_0 (the arrow denotes a jump from a point with ordinate z_0 to a point with ordinate y_0 such that $|z_0 - y_0| = L_1$).

If one now chooses a point with ordinate y_1, where $y_1 = y(0, \xi_1)$, $\xi_0 < \xi_1$ and $y_0 < y_1$, as the initial state in the neighbourhood of y_0, it follows from Lemma 1.2 that the sequences of points $(y_k)_{k \geq 1}$, $(z_k)_{k \geq 1}$, which correspond to the position of the representative points after each jump along the ordinate axes, will satisfy the chains of inequalities

$$y_1 > y_3 > y_5 > \ldots > y_0 > \ldots > y_6 > y_4 > y_2$$

$$z_2 > z_4 > z_6 > \ldots > z_0 > \ldots > z_5 > z_3 > z_1 \qquad (3.1)$$

If, using the same notation, $y_1 < y_0$, we shall have chains of inequalities which differ from (3.1) by interchange of the symbols $y \leftrightarrow z$.

Hence, for any initial state of a representative point in the neighbourhood of y_0, the sequence of its ordinates on the line $x = 0$ monotonically approaches y_0 from two sides on the positive semi-axes and z_0 on the negative semi-axis of the same axis. It can be shown in a similar way to the result obtained previously ([6, Theorem 1]; see also [7]) that such a monotonic approach will be an asymptotic convergence to a limiting cycle with a single jump in the half-period of the oscillation period of the oscillator.

The following theorem therefore holds.

Theorem 3. For any value $L_1 > 0$, system (2.1) has a unique orbitally asymptotically stable periodic motion with a single jump per half-period of the oscillation period of the oscillator.

Remark 3.1. The attraction domain of the limiting cycle of system (2.1) with a single jump in a half-period of the oscillation period of the oscillator is completely determined by the behaviour, when $t \to \pm\infty$, of the trajectories in the neighbourhood of the trajectory which "enters" the stationary point after a finite time. This trajectory always exists in the case of (2.1) and has no analogue in the theory of continuous dynamical systems. This domain can be both finite and also contain points located in an infinite part of the phase plane.

4. A TRAJECTORY "ENTERING" A STATIONARY POINT AFTER A FINITE TIME

It is well known that, in a continuous dynamical system with the property of uniqueness of the solutions, a representative point, moving along trajectories of the system, can "enter" the stationary point from any regular point of the phase plane only when $t \to +\infty$ or $t \to -\infty$. As far as system (2.1) is concerned, it always has a trajectory which "enters" the stationary point after a finite time (after a finite number of jumps). Actually, the trajectory which reaches the negative semi-axis of the ordinate axis at a point, the ordinate of which is equal to $-L_1$, is the trajectory of system (2.1) which "enters" the stationary point after a finite time.

We will now show how the attraction domain G of the limiting cycle γ with a single jump in a half-period of the oscillation period of the oscillator changes, depending on the location of the trajectory Γ which "enters" the stationary point of system (2.1). We will use the notation $y_{-L_1} = y(0, \xi_{-L_1})$ where the value of y_{-L_1} is always greater than L_1.

Theorem 4.1. If $y_{-L_1} > L_2$, then $G = R^2 \backslash \Gamma$, where Γ has points outside a circle of any radius with its centre at the stationary point.

Proof. We construct sequences of intervals I_k and J_k ($k = 1, 2, \ldots$) on the ordinate axes when $y > 0$ and $y < 0$, respectively, in the following way. We put $I_1 = (L_2, y_{-L_1})$, $I_2 = (0, y_{-L_1} - L_2)$. Then, on taking account of the fact that the lengths of the intervals I_1 and I_2 are equal, we arrive at the conclusion that the representative points which move, as the time t increases, in the half-plane $x < 0$ along trajectories which border on points of the interval J_2, find themselves, after reaching the ordinate axes and jumping upwards by an amount L_2, at points of the interval I_1. If, however, one considers the motion of representative points which have been noted in the reverse direction, then a single-valued mapping of the interval $I_1 \to I_2$ will correspond to such a motion of these points for which points of the interval I_2, which do not belong to the trajectories of (2.1) will correspond to points of the interval I_1, which do belong to the trajectories of system (2.1). Now, on noting that the trajectories of system (2.1), which leave from points of the interval $J_2 = (z^2, 0)$ of the ordinate axis, where $z^{(2)} = z(0, \eta_2)$, are adjacent to points of the interval I_2 by Lemma (1.2) we arrive at the conclusion that to points of the interval I_2, by Lemma (1.2) we arrive at the conclusion that $|J_2| > |I_2|$. Here, after a jump upwards by an amount L_1, the representative points of system (2.1) find themselves in the interval J_2. It is therefore possible to speak of a mapping of the intervals J_2 and J_3 which are equal in length, where $J_3 = (z^{(2)} - L_1, L_1)$. In the case of such a mapping, points in the interval J_3 which do not belong to the trajectories of the system being considered, correspond to points in the interval J_2, which do belong to the trajectories of system (2.1). Furthermore, the interval $I_3 = (y_{-L_1}, y_{-(z^{(2)}-L_1)})$ corresponds to the interval J_3 such that every point belonging to the interval I_3, on starting to move along the trajectories of system (2.1) in the half-plane $x > 0$ as the time t increases, will be adjacent, at a certain instant of time, to a point of the interval J_3. Using this type of motion, the one-to-one mapping of the intervals J_3 and I_3 is determined and, by Lemma 1.2, the inequality $|J_3| > |I_3|$ will hold.

Using similar reasoning, sequences of intervals I_k, J_k $(k = 1, 2, \ldots)$ can be constructed such that

$$|I_1|=|I_2|<|J_2|=|J_3|<|I_3|=|I_4|<|J_4|=|J_5|< \ldots \tag{4.1}$$

These intervals fill the whole of the ordinate axis with the exception of a denumerable set of points which correspond to the ends of the intervals and where intervals with even subscripts contain the points of the trajectory of system (2.1), while the points of the intervals with odd subscripts do not belong to the trajectories.

When account is taken of what has been said, relations (4.1) mean that the attraction domain G of the limiting cycle γ coincides with the whole phase space, with the exception of trajectories Γ having points outside a circle of arbitrary radius with its centre at the stationary point, any representative point of which, after a finite time, occurs at the origin of the system of coordinates. The theorem is proved.

As far as the cases $y_{-L_1} = L_2$ and $y_{-L_1} < L_2$ are concerned, when the inequality $y_{-L_1} < L_2$ is taken into account, we also conclude, by Lemma 1.2, that the following two assertions hold.

Theorem 4.2. If $y_{-L_1} = L_2$, the attraction domain G of the limiting cycle γ is finite and bounded by the contour

$$(L_2, \xi_{-L_1}, -L_1 \xrightarrow{L_1} 0, L_2).$$

Each representative point outside the domain G asymptotically approaches a finite trajectory Γ when $t \to +\infty$.

Theorem 4.3. If $y_{-L_1} < L_2$, the attraction domain G of the limiting cycle γ is finite and bounded by the contour

$$(y_{-L_1}, \xi_{-L_1}, -L_1 \xrightarrow{L_1} 0, y_{-L_1}).$$

Each representative point outside the domain G, lying in a sufficiently small external half-neighbourhood of Γ, asymptotically approaches a finite trajectory Γ when $t \to -\infty$.

Remark 4. It follows from Theorems 4.1–4.3 that system (2.1) can execute periodic oscillations with two jumps in a period solely in the case when $y_{-L_1} < L_2$

A geometrical interpretation in the plane of the parameters L_1 and L_2 can be given (Fig. 1) for all situations described by the last three theorems. Here, the domain between the half-line $L_1 = L_2$ and the curve $L_2 = y_{-L_1}$ corresponds to the case described in Theorem 4.1. The points of the curve $L_2 = y_{-L_1}$ correspond to the case described in Theorem 4.2. Finally, the domain between the positive semi-axis $L_1 = 0$ and the curve $L_2 = y_{-L_1}$ (the hatched area) corresponds to the case described in Theorem 4.3.

5. PERIODIC OSCILLATIONS WITH TWO JUMPS IN A PERIOD

Suppose that a representative point which leaves from the point y_+ goes around the contour

$$(y_+, \xi, z_+ \xrightarrow{L_1} z_-, \eta, y_-) \tag{5.1}$$

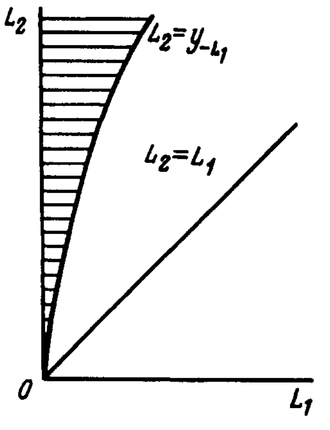

Fig. 1.

where $y_+ = y(0, \xi)$, $z_+ = z(0, \xi)$, $z_- = z(0, \eta)$, $y_- = y(0, \eta)$ along trajectories of system (2.1) and, as previously, the arrow indicates a jump from the point z_+ to the point z_- at a distance equal to L_1.

We shall call such a contour a link which is obviously defined by just a single parameter $\xi > \xi_{-L_1}$.

We put $Y(\xi) = y_+ - y_- \equiv y(0, \xi) - y(0, \eta(\xi))$. Then, if, for a certain value of $\xi = \xi^* > \xi_{-L_1}$, the magnitude of the jump L_2 on the positive semi-axis of the ordinate axis coincides with $Y(\xi)^*$, then the link closes itself and will correspond to a periodic self-excited oscillatory motion of system (2.1) with two jumps in a period.

Under a certain condition such a closed link always exists and is unique.

In fact, suppose that the link (5.1) corresponds to the point $\xi > \xi_{-L_1}$ and that the link

$$(\bar{y}_+,\ \bar{\xi},\ \bar{z}_+ \xrightarrow{L_1} \bar{z}_-,\ \bar{\eta},\ \bar{y}_-)$$

corresponds to the point $\bar{\xi} < \xi$.

We will now estimate the increment $\Delta Y = Y(\xi) - Y(\bar{\xi})$. Using Lemma 1.2, we have

$$\Delta Y > (y_+ - \bar{y}_+) - (\bar{z}_- - z_-) > 0$$

since $\bar{z}_- - z_- = \bar{z}_+ - z_+$.

Hence, $Y(\xi) > Y(\bar{\xi})$ when $\xi > \bar{\xi}$, that is, $Y(\xi)$ is a function which increases strictly monotonically. This fact, the equality $Y(\xi_{-L_1}) = y(0, \xi_{-L_1})$ and arguments, analogous to those used in proving the orbital asymptotic stability of a periodic motion with a single jump in a half-period of the oscillation period of the oscillator, lead to the conclusion that the following theorem holds.

Theorem 5.1. The condition

$$y_{-L_1} < L_2 < \sup_{\xi > \xi_{-L_1}} Y(\xi)$$

is necessary and sufficient for system (2.1) to have a unique orbitally asymptotically stable periodic motion with two jumps in a period corresponding to the periodic self-excited oscillations of the dynamic clock model.

Remark 5.1. A periodic motion with two jumps in a period always exists if the difference $L_2 - y(0, \xi_{-L_1})$ is sufficiently small.

Remark 5.2. It is obvious that

$$y_{-L_1} = y(0,\ z^{-1}(-L_1))$$

where z^{-1} is a mapping which is the inverse of z. The value of y_{-L_1} can also be found as a root of the equation $x(y(0, \xi_{-L_1})) = 0$, where x is the solution of the equation

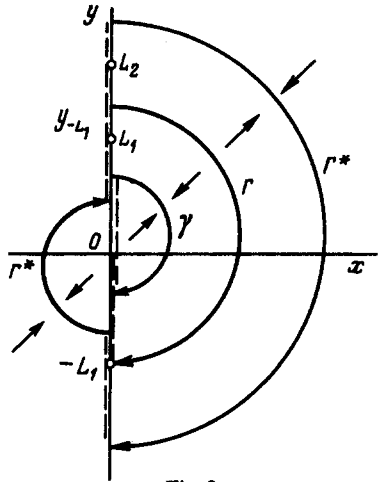

Fig. 2.

$$dx/dy = -y/(g(x)+f(x)y)$$

which satisfies the initial condition $x(0) = -L_1$.

Theorem 5.2. The period of the self-excited oscillations of the dynamic clock model (2.1) is identical with the period of the isochronous oscillations of the Liénard oscillator (1.1).

The validity of the theorem follows from the property of strong isochronism of the oscillations of the oscillator (1.1) and the instantaneous nature of the impulses which are transmitted by the escapement to the oscillator.

Remark 5.3. When the conditions of Theorem 5.1 are satisfied, the mutual arrangement of the limiting cycle with a single jump in a half-period of the oscillation of the oscillator γ, of the limiting cycle with two jumps in a period Γ^* and the trajectory which "enters" the stationary point after a finite time, Γ, is shown in Fig. 2.

Remark 5.4. If a model which is described by system (2.1) with a strongly isochronous periodic Liénard oscillator (in this case the function $f(x)$ is odd) is chosen as the dynamic model of a clock, then Theorems 3.1, 4.1, 4.2, 4.3, 5.1 and 5.2 also hold for such a model.

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